The de Rham Witt complex and crystalline cohomology

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If X/k is a smooth projective scheme over a perfect field k, let us try to find an explicit quasi-isomorphism $Ru_{X/W*}(\mathcal{O}_{X/S}) \cong \mathcal{W}\Omega^{\cdot}_{X}$. [1] To do this we need an explicit representative of $Ru_{X/W*}(\mathcal{O}_{X/S})$ together with its Frobenius action. The standard way to do this is to choose an embedding $X \to \tilde{Y}$, where $\tilde{Y}/W$ is smooth and endowed with a lift $\phi_{\tilde{Y}}$ of Frobenius. For example, if X is quasi-projective, we can let $\tilde{Y}$ be a projective space $\mathbf{P}^N$ endowed with the endomorphism defined by raising coordinates to their pth power. (If X is not projective, one can use local liftings and simplicial methods; which we shall not discuss here.) Once such an embedding $(\tilde{Y}, \phi_{\tilde{Y}})$ is chosen, let Y be its reduction modulo p, let $\tilde{D}$ denote the (p-adically complete) PD-envelope of X in $\tilde{Y}$ and let D be its reduction modulo p. Then the $\mathcal{O}_{\tilde{Y}}$-module $\mathcal{O}_{\tilde{D}}$ admits an integrable connection [1, 6.4], whose corresponding de Rham complex $\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}$ is a representative of $Ru_{X/W*}(\mathcal{O}_{X/W})$ [1, 7.1]. The assumed lifting $\phi_{\tilde{Y}}$ of F_Y extends uniquely to a PD-morphism $\phi_{\tilde{D}}$ of $\tilde{D}$. This morphism induces an endomorphism $\phi^{\cdot}_{\tilde{D}}$ of $\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}$. Since X and $\tilde{Y}$ are smooth, the terms of this complex are p-torsion free and p-adically complete. The endomorphism $\phi^{\cdot}_{\tilde{D}}$ is $\mathcal{O}_{\tilde{D}}$-linear and ϕ_D vanishes on $\Omega^1_{Y/k}$, hence $\phi^1_{\tilde{D}}$ is divisible by p and $\phi^i_{\tilde{D}} = p^i F$ for a unique $\mathcal{O}_{\tilde{D}}$-linear endomorphism F of $\mathcal{O}_{\tilde{D}} \otimes \Omega^i_{\tilde{Y}/W}$. Thus $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d, F)$ is a Dieudonné complex.

[1] It is hard to find an explicit construction of this isomorphism in [2], although of course it does follow from the comparison [2, 4.4.12] with the classical de Rham Witt complex and Illusie's theorem [II 1.4][3]. In fact Illusie explains a new version of the proof in his note [4]. The method presented here different. I would like to thank Illusie for very helpful conversations concerning it.

Theorem 1 *Let X/k be a smooth scheme over a perfect field k, embedded as a locally closed subscheme of a smooth $\tilde{Y}/W$ which is endowed with a lifting $\phi_{\tilde{Y}}$ of the Frobenius endomorphism of its reduction Y/k modulo p. Then the Dieudonné complex $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d, F)$ constructed above is in fact a (torsion-free) Dieudonné algebra. Moreover, the natural map*

$$(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d) \to \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d)$$

is a quasi-isomorphism, and the natural map

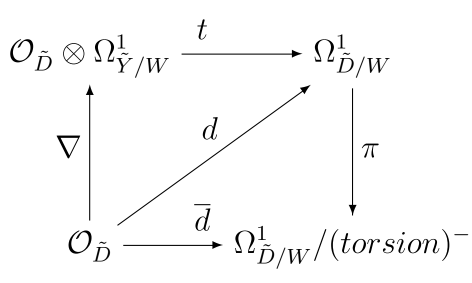
$$\mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d, F) \to (\mathcal{W}\Omega^{\cdot}_X, d, F)$$

is an isomorphism. Thus, $(\mathcal{W}\Omega^{\cdot}_X, d)$ is a representative of $Ru_{X/W*}(\mathcal{O}_{X/W})$.

Proof: To see that $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, D, F)$ is a Dieudonné algebra, we must show that $\phi_{\tilde{D}}: \tilde{D} \to \tilde{D}$ reduces to the Frobenius endomorphism F_D of D [2, 3.1.2]. The reduction ϕ_D of $\phi_{\tilde{D}}$ is the unique PD morphism $D \to D$ extending F_Y, and so it will suffice to show that F_D is in fact a PD-morphism. But if t is an element of the PD-ideal $\overline{I}$ of X in D, then $F_D^*(t) = t^p = p!t^{[p]} = 0$, and hence for any $n \geq 1$, $F_D \circ \gamma_n = \gamma_n \circ F_D = 0$.

Note that $\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}$ is not the same as the the de Rham complex of $\tilde{D}$; the latter has a lot of p-torsion.

Lemma 2 *In the following diagram, the lower triangle commutes, even though the upper one does not. (NB: here we always mean the p-adically completed de Rham complexes; and in particular we are dividing by the p-adic closure of the torsion in the lower right hand corner.)*



Furthermore, the composite

$$\bar{t}: \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{\tilde{Y}/W} \longrightarrow \Omega^1_{\tilde{D}/W}/(torsion)^-$$

is an isomorphism.

Proof: The top horizontal arrow in the diagram is induced by adjunction. The algebra $\mathcal{O}_D$ is locally generated over $\mathcal{O}_{\tilde{Y}}$ by the divided powers $f^{[n]}$ of elements f of the ideal of X in $\tilde{Y}$, for $n \geq 1$. For any such f, we have $\nabla f^{[n]} = f^{[n-1]} \otimes df$ in $\mathcal{O}_{\tilde{D}} \otimes \Omega^1_{\tilde{Y}/W}$ [1, 6.4]. On the other hand, since $n! f^{[n]} = f^n$ and $n! f^{[n-1]} = n f^{n-1}$, we have

$$n! df^{[n]} = d(n! f^{[n]}) = df^n = n f^{n-1} df = n(n-1)! f^{[n-1]} df = n! f^{[n-1]} df$$

in $\Omega^1_{\tilde{D}/W}$. Thus $\nabla f^{[n]}$ and $df^{[n]}$ have the same image in $\Omega^1_{\tilde{D}/W}/(torsion)$, so the lower triangle commutes.

Since $d \colon \mathcal{O}_{\tilde{D}} \to \Omega^1_{\tilde{D}/W}$ is the universal derivation to a p-adically complete sheaf of $\mathcal{O}_{\tilde{D}}$-modules, there is a unique map $s \colon \Omega^1_{\tilde{D}/W} \to \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{\tilde{Y}}$ such that $s \circ d = \nabla$; this map factors through a map

$$\overline{s} \colon \Omega^1_{\tilde{D}/W}/(torsion)^- \to \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{\tilde{Y}}.$$

Then

$$\overline{t} \circ s \circ d = \pi \circ t \circ s \circ d = \pi \circ t \circ \nabla = \pi \circ d,$$

and it follows that $\overline{t} \circ s = \pi$ and hence that $\overline{t} \circ \overline{s} = \mathrm{id}$. On the other hand, if f is a local section of $\mathcal{O}_{\tilde{Y}}$, then f can also be viewed as a section of $\mathcal{O}_{\tilde{D}}$, and $\nabla f = 1 \otimes df$ in $\mathcal{O}_{\tilde{D}} \otimes \Omega^1_{\tilde{Y}/W}$. Thus the upper right triangle of the diagram does commute when restricted to $\mathcal{O}_{\tilde{Y}}$, and it follows that $\overline{s} \circ \overline{t} = \mathrm{id}$. □

Since $(\tilde{D}, \phi_{\tilde{D}})$ is a p-torsion free lifting of (D, F_D) so by [2, 3.2.1], there is an endomorphism F of the graded abelian sheaf $\Omega^{\cdot}_{\tilde{D}/W}$ which gives it the structure of a Dieudonné algebra.

Lemma 3 *The map t in Lemma 2 induces an isomorphism of Dieudonné algebras:*

$$w \colon \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}, d, F) \to \mathcal{W}Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F).$$

Proof: By construction, the natural map

$$Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F) \to Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F)/(torsion).$$

is an isomorphism, and hence so is the natural map

$$\mathcal{W}Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F) \to \mathcal{W}Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F)/(torsion)^-.$$

since both are p-adically complete. Thus the result follows from the second statement of Lemma 2. □

Since $(\tilde{D}, \phi_{\tilde{D}})$ is a p-torsion free lifting of (D, F_D), [2, 4.2.3] implies that there is an isomorphism of Dieudonné algebras:

$$\mathcal{W}Sat(\Omega^{\cdot}_{\tilde{D}/W}, d, F) \to (\mathcal{W}\Omega^{\cdot}_D, d.F).$$

Thus we find a commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W} & \xrightarrow{\bar{t}} & \Omega^{\cdot}_{\tilde{D}/W}/(torsion)^{-} & & \\
\downarrow s & & \downarrow & & \\
\mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}) & \xrightarrow{w} & \mathcal{W}Sat(\Omega^{\cdot}_{\tilde{D}/W}) & \xrightarrow{\cong} & \mathcal{W}\Omega^{\cdot}_D \\
 & & & \swarrow g & \\
 & & \mathcal{W}\Omega^{\cdot}_X & &
\end{array}$$

We have seen that $\bar{t}$ and w are isomorphisms. Since X is the reduced subscheme of D, the following lemma, which is a consequence of [2, 6.5.2] and also of the easier [2, 36.1], implies that g is also an isomorphism.

Lemma 4 *If Z is scheme over $\mathbf{F}_p$, the natural map $Z_{red} \to Z$ induces an isomorphism $\mathcal{W}\Omega^{\cdot}_Z \to \mathcal{W}\Omega^{\cdot}_{Z_{red}}$.* □

We conclude that the natural map $\mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}) \to \mathcal{W}\Omega^{\cdot}_X$ is an isomorphism, as asserted in the second statement of Theorem 1.

It remains to prove that the map $\mathcal{O}_{\tilde{D}} \otimes^{\cdot}_{\tilde{Y}/W} \to \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W})$ is a quasi-isomorphism. The complex $\mathcal{O}_{\tilde{D}} \otimes^{\cdot}_{\tilde{Y}/W}$ is not of Cartier type, and I could not find a direct reference in [2] which proves this. But it suffices to copy some of its arguments. By [1, 8.20], applied to the constant gauge $\epsilon = 0$, the morphism $\phi^{\cdot}_{\tilde{D}}$ factors through a quasi-isomorphism

$$\alpha \colon \mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W} \to \eta_p(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}).$$

It follows that $\eta_p^n(\alpha)$ is a quasi-isomorphism for every n, and hence that the map

$$\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W} \to \varinjlim \eta_p^n(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}) = Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W})$$

is also a quasi-isomorphism. Since both complexes are p-torsion free, this map remains a quasi-isomorphism when reduced modulo p^n for every n, and by [2, 2.8.1], the map

$$Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}) \otimes \mathbf{Z}/p^n\mathbf{Z} \to \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W}) \otimes \mathbf{Z}/p^n\mathbf{Z}$$

is also a quasi-isomorphism for every n. We conclude that the map

$$\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W} \to \mathcal{W}Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{\tilde{Y}/W})$$

is a quasi-isomorphism when reduced modulo p. Since both sides are p-adically complete and p-torsion free, it follows that it too is a quasi-isomorphism. (not safe to pass to limit). □

References

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